

Generalized Uncertainty Principle and Klein Paradox

Sumit Ghosh *

S. N. Bose National Centre for Basic Sciences
Block JD, Sector III, Salt Lake, Kolkata 7000098

February 10, 2012

Abstract

We have studied the Klein paradox in presence of a new generalized uncertainty principle. We define a new momentum operator and derive the modified dispersion relation for a Dirac particle. Apart from the forbidden band within the range $\pm m$, m being the mass of the particle, we find the existence of additional forbidden bands which cause the allowed energy bands to be localised within finite energy range. This band structure forbids a Dirac particle from penetrating a potential barrier of sufficient height which is estimated to be of the order of Planck energy (E_P). This is also true for a massless particle. It reflects from the wall of a sufficiently high ($\sim E_P$) potential like a massive particle does in nonrelativistic quantum mechanics. We propose this to be a signature of the existence of a minimum length and a maximum energy.

1 Introduction

Introduction of a minimum length scale [1]-[7] has revealed several salient features of gravity and quantum field theory. The essence of a minimum length can be captured by various means - by a generalized uncertainty principle (GUP), by a modified dispersion relation (MDR), by a deformed special relativity (DSR) etc. One can initially start from a modified momentum operator and successively obtain a GUP or an MDR [8]. The manifestation of the presence of a minimal length in different fields, such as black hole thermodynamics [9]-[12], non relativistic and relativistic quantum mechanics [13]-[19] etc, has stimulated a large number of extensive studies.

In this letter we study the Klein paradox in presence of a new GUP which imply a minimum length as well as a maximum energy. We start from a modified momentum operator and derive the MDR for a Dirac particle. We show that apart from the forbidden energy band in between $\pm m$ (m is the mass of the particle) as predicted by standard relativistic quantum mechanics, there are other forbidden regions at some high (both positive and negative) energies which are of the order of Planck energy (E_P). The wave vector, after certain value of energy, instead of increasing monotonically starts decreasing with an increase of energy. This is consistent with the existence of a minimum length scale. The allowed energy band is localised within finite energy range which give rise to interesting features of the Klein paradox. We further study the case of a massless particle and find an interesting result. According to non-GUP relativistic quantum mechanics, a massless Dirac particle can penetrate a potential barrier of any arbitrary height. In presence of our GUP the situation is entirely different from that. We find that a massless particle can not penetrate a potential barrier of the order of E_P which we propose to be a signature of the existence of minimum length and maximum energy.

The organisation of letter is as follows. In section 2, We introduce A modified momentum operator and obtain the corresponding Dirac equation. In section 3, we find the solution for the modified Dirac equation and derive the MDR which is used to find the band structure. This band structure is used to explain the nature of the Klein paradox in section 4. In section 5 we investigate the scenario for a massless particle. Section 6 contains the discussions.

2 GUP and the Dirac equation

In this section we will derive a new GUP and obtain corresponding modified Dirac equation. In a recent paper [12] we have proposed a GUP based on some heuristic arguments. The GUP contains only even terms in right hand side. This is needed for our prescription, but not explicitly necessary for a GUP. In this letter we will use a more general form which contains both odd and even terms.

*E-mail: sumit.ghosh@bose.res.in

This kind of GUP is recently proposed in [14]. It not only accommodate a minimum length, but also a maximum measurable momentum (and hence a maximum energy). We will show a systematic derivation of the GUP starting from a modified momentum operator and consecutively will derive the Dirac equation in presence of such a GUP.

For simplicity we will restrict ourselves within one dimension. We define a new momentum operator(P) by keeping first two correction terms to the order of E_P^{-2}

$$P = p \left[1 - a_0 \frac{p}{E_P/c} + b_0 \left(\frac{p}{E_P/c} \right)^2 \right] \quad (1)$$

where E_P is the Planck energy. a_0 and b_0 are real positive numbers and p obeys the standard commutation relation

$$[x, p] = i\hbar \quad (2)$$

x being the coordinate operator.

It would be convenient for future calculations to introduce the scaled coefficients

$$a = \frac{a_0}{E_P/c} \quad b = \frac{b_0}{(E_P/c)^2} \quad (3)$$

and express (1) in terms of them. The modified momentum operator can be recast as

$$P = p(1 - ap + bp^2) \quad (4)$$

We define the new coordinate operator as

$$X = x \quad (5)$$

The GUP in terms of the new coordinate and momentum is given by

$$\begin{aligned} \Delta X \Delta P &\geq \left| \frac{1}{2} \langle [X, P] \rangle \right| \\ &\geq \frac{\hbar}{2} (1 - 2aP + (3b - 2a^2)P^2) \end{aligned} \quad (6)$$

The existence of minimum position uncertainty requires the coefficient of P^2 to be positive. i.e

$$3b - 2a^2 > 0 \quad (7)$$

Note that by putting $b = 2a^2$ we can obtain the GUP proposed in [14] and this is also consistent with the above condition (7).

The Dirac equation in presence of this modified momentum operator (4) is

$$H\psi = [c(\vec{\alpha} \cdot \vec{p})(1 - a(\vec{\alpha} \cdot \vec{p}) + b(\vec{\alpha} \cdot \vec{p})(\vec{\alpha} \cdot \vec{p})) + \beta mc^2] \psi \quad (8)$$

where α and β are the Dirac matrices defined as

$$\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \quad (9)$$

σ_i is the i th Pauli matrix.

For simplicity we will consider unidirectional motion (say along z axis). The one dimensional Dirac equation for a time independent Hamiltonian is

$$[c\alpha_z p - cap^2 + cb\alpha_z p^3 + \beta mc^2] \psi = E_G \psi \quad (10)$$

where E_G is the GUP corrected energy of the system. For future use we will replace the momentum operator p by its differential form $-i\hbar \frac{d}{dz}$ and recast the modified Dirac equation as

$$\left[-i\hbar c\alpha_z \frac{d}{dz} + \hbar^2 ca \frac{d^2}{dz^2} - i\hbar cb\alpha_z \frac{d^3}{dz^3} + \beta mc^2 \right] \psi = E_G \psi \quad (11)$$

3 Solution of modified Dirac equation for a free particle and its band structure

In this section we will find a solution of the modified Dirac equation (11) and subsequently derive the MDR which we will use to find the energy band structure.

The one dimensional (along z direction) Dirac equation for a free particle without any GUP correction is

$$\left(-i\hbar c\alpha_z \frac{d}{dz} + \beta mc^2\right) \psi(z) = E\psi(z) \quad (12)$$

One can obtain this equation by putting $a = b = 0$ in (11). Its positive energy (+E) solution is given by

$$\psi(z) = Ne^{ikz} \begin{pmatrix} \chi \\ \mathcal{U}\sigma_z\chi \end{pmatrix} \quad (13)$$

where χ is a (2×1) column matrix which satisfies orthonormality condition $\chi^\dagger\chi = 1$. N is the normalization factor. \mathcal{U} is a dimensionless quantity defined as

$$\mathcal{U} = \frac{c\hbar k}{E + mc^2} \quad (14)$$

k is the wave number which obeys the dispersion relation

$$E = \pm \sqrt{(c\hbar k)^2 + m^2 c^4} \quad (15)$$

For the modified Dirac equation, the solution would not be so simple. Whatever the solution be, it must converge to (13) in absence of GUP. We can consider the following ansatz

$$\psi_G(z) = Ne^{ik_G z} \begin{pmatrix} \chi \\ \mathcal{U}_G\sigma_z\chi \end{pmatrix} \quad (16)$$

to be a solution of (11).

The assumption

$$\lim_{(a,b) \rightarrow 0} \psi_G \rightarrow \psi \quad (17)$$

where ψ is given by (13), requires

$$\lim_{(a,b) \rightarrow 0} \mathcal{U}_G \rightarrow \mathcal{U} \quad (18)$$

where \mathcal{U} is given by (14). Besides, the dispersion relation also should converge to (15). i.e.

$$\lim_{(a,b) \rightarrow 0} E_G \rightarrow \pm \sqrt{(c\hbar k_G)^2 + m^2 c^4} \quad (19)$$

Using the ansatz (16) in (11) we can obtain the GUP corrected energy (E_G) or the MDR and the expression for \mathcal{U}_G which are given by

$$E_G = \pm \sqrt{c^2 \hbar^2 k_G^2 (1 + b\hbar^2 k_G^2)^2 + m^2 c^4 - ca\hbar^2 k_G^2} \quad (20)$$

$$\mathcal{U}_G = \frac{c\hbar k_G (1 + b\hbar^2 k_G^2)}{ca\hbar^2 k_G^2 + mc^2 + E_G} \quad (21)$$

One can readily see that both E_G and \mathcal{U}_G satisfy (19) and (18).

A necessary condition for the existence of a positive energy solution is

$$(a^2 - 2b) < \left(\frac{1}{(\hbar k_G)^2} + \frac{m^2 c^2}{(\hbar k_G)^4} + b^2 (\hbar k_G)^2 \right) \quad (22)$$

If we consider particles for which $\hbar k \gg m$ and $0 < b(\hbar k_G)^2 \ll 1$ then the condition reduces to

$$(a^2 - 2b) \lesssim 0 \quad (23)$$

Notice that in [14], $b = 2a$ which also satisfies this condition. We can refine this condition by using (7), from which one can easily obtain

$$a^2 - 2b < -b/2 \quad (24)$$

Since the coefficient b is positive, from (24) we get

$$(a^2 - 2b) < 0 \quad (25)$$

Therefore the existence of a minimum length scale also asserts the existence of a positive energy solution.

The expression for the wave vector k_G can be recovered from (20) by ignoring $\mathcal{O}(b^2)$ term as

$$\hbar^2 k_G^2 = \frac{(c - 2aE_G)}{2(a^2 - 2b)c} \left[1 \pm \sqrt{1 - \frac{4(a^2 - 2b)(E_G^2 - m^2 c^4)}{(c - 2aE_G)^2}} \right] \quad (26)$$

We can further simplify the expression by expanding the square root part. The simplified expression for the wave vector to $\mathcal{O}(a^2, b)$ is

$$\hbar^2 k_G^2 = \frac{E_G^2 - m^2 c^4}{c^2 - 2aE_G c} \left[1 + \frac{(a^2 - 2b)(E_G^2 - m^2 c^4)}{4(c - 2aE_G)^2} \right] \quad (27)$$

Note that here we have considered only negative (-) part of the square root term, because it is consistent with the fact that at the limit $(a, b) \rightarrow 0$ it gives

$$\lim_{(a,b) \rightarrow 0} \hbar^2 k_G^2 = E^2 - m^2 c^4 \quad (28)$$

where we used the fact that $\lim_{(a,b) \rightarrow 0} E_G = E$.

The above expression (27) shows an interesting property of a free GUP Dirac particle. As the particle energy E_G approaches E_P the denominator

$$(c^2 - 2aE_G c) = c^2 \left(1 - 2a_0 \frac{E_G}{E_P} \right) \quad (29)$$

where we have used (3), approaches zero which will cause the wave vector k_G to diverge. This contradicts with the basic assumption of the existence of minimum length. Hence it gives an upper limit for the measurable energy which is given by

$$E_G|_{max} = \frac{E_P}{2a_0} \quad (30)$$

This is a direct indication of the existence of maximum energy and hence a maximum momentum. Note that the order of maximum energy matches well with that predicted in [14]. In this letter we will limit ourselves within energy less than this maximum energy (30). We will show soon, before reaching the energy value the wave vector will be imaginary and the particle will encounter a forbidden energy band (*fig.1*). Hence there will not be any propagating mode in the vicinity of this maximum energy.

We will now study the band structure. The simplest way to know the occurrence of a propagative or non-propagative mode is to find the nature of k_G^2 with respect to energy (E_G). A negative value of k_G^2 suggests a imaginary k_G which is a damped mode. Corresponding energy values form a forbidden band. (*fig.1*) shows the band structure for the modified Dirac equation. The forbidden band within the range $\pm m$ is well known in Dirac theory without GUP. The interesting feature is the presence of two more forbidden bands which are the sole effect of the GUP. The edges of the allowed and forbidden bands, which are actually zeros of (27), are given by

$$W_G^1 = \frac{\sqrt{c^2(a^2 - 2b)(c^2 m^2(17a^2 - 2b) - 4)} + 8ac}{17a^2 - 2b} \quad (31)$$

$$W_G^2 = -m \quad (32)$$

$$W_G^3 = m \quad (33)$$

$$W_G^4 = \frac{\sqrt{c^2(a^2 - 2b)(c^2 m^2(17a^2 - 2b) - 4)} - 8ac}{17a^2 - 2b} \quad (34)$$

The edge W_G^1 is the bottom of the Dirac sea, whereas W_G^4 gives the maximum positive energy for a free particle. Based on these values we can divide the entire energy spectrum in five different regions which are described in (*Table1*).

FB1	1st forbidden band	$E_G < W_G^1$
AB1	1st allowed band	$W_G^2 < E_G < W_G^1$
FB2	2nd forbidden band	$W_G^3 < E_G < W_G^2$
AB2	2nd Allowed band	$W_G^4 < E_G < W_G^3$
AB3	3rd forbidden band	$W_G^4 < E_G$

Table 1: Nomenclature and positions of different bands

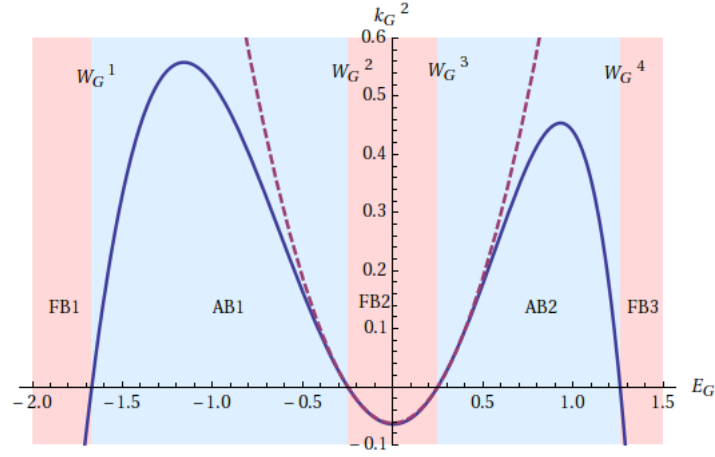


Figure 1: k_G^2 as a function of energy E_G for $(a = 0.05, b = 1, m = 0.25)$ with $c = \hbar = E_P = 1$. The dashed line shows non GUP dispersion relation $k^2 = E^2 - m^2$. The light-red regions (FB1, FB2, FB3) are forbidden bands and the light-blue regions (AB1, AB2) are the allowed bands.

4 GUP and Klein paradox

In this section we will study the effect of GUP on Klein paradox. Klein paradox is a well known phenomenon for a massive Dirac equation. The motion of a plane wave inside a step potential of height V_0 with energy (E) less than V_0 is well known in nonrelativistic quantum mechanics and we know that the wave will decay exponentially with an exponent $\kappa = \sqrt{2m(V_0 - E)}$. The expression suggests that if we increase V_0 the wave will die off faster and so long $E < V_0$ there will not be any propagative mode. In case of a Dirac particle one can easily check that if the barrier height $V_0 > 2m$ then there can be a propagative mode for $E < V_0$ and the wave can penetrate through the barrier. Let us see how the effect is modified in presence of a GUP.

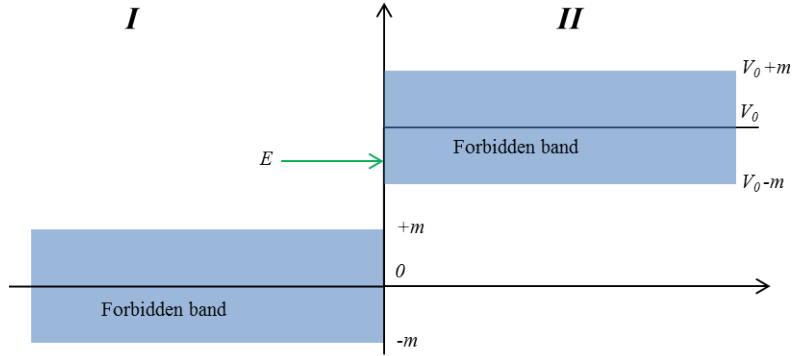


Figure 2: A free particle with energy E facing a potential step of height V_0 ($V_0 > E$). The shaded regions are forbidden bands. A particle can not penetrate region (II) if it encounter a forbidden band there.

Consider the situation when a particle with positive energy and up spin is incident on the potential step from the left. The solution for the incident wave in region (I) is (apart from the normalization factor)

$$\psi_{in}^G(z) = e^{ik_G z} \begin{pmatrix} \chi \\ \mathcal{U}_G \sigma_z \chi \end{pmatrix} \quad (35)$$

where \mathcal{U}_G and k_G are given by (21) and (27) respectively.

Let us look for the solution in region II (fig.2). The transmitted part will only contain the spin up component and the solution can be given by the ansatz

$$\psi_{tr}^G = e^{iq_G z} \begin{pmatrix} \chi \\ \mathcal{U}'_G \sigma_z \chi \end{pmatrix} \quad (36)$$

where q_G and \mathcal{U}'_G is given by

$$\hbar^2 q_G^2 = \frac{(E_G - V_0)^2 - m^2 c^4}{c^2 - 2a(E_G - V_0)c} \left[1 + \frac{(a^2 - 2b)((E_G - V_0)^2 - m^2 c^4)}{4(c - 2a(E_G - V_0))^2} \right] \quad (37)$$

$$\mathcal{U}'_G = \frac{c\hbar q_G(1 + b\hbar q_G^2)}{ca\hbar^2 q_G^2 + mc^2 + (E_G - V_0)} \quad (38)$$

Let consider a particle with such a energy that in region (II) it falls in a forbidden energy band (i.e. its momentum will be imaginary and the particle can not propagate). By gradually increasing the potential energy we can lift up the negative energy bands such a way that the particle will find itself in an allowed band. Its momentum will be a real quantity and it can propagate inside the potential step. This is the physical scenario behind the Klein paradox. In a non-GUP Dirac theory, if we keep increasing the barrier height arbitrarily, there will always be room for propagating modes in region (II). In presence of a GUP the picture is entirely different. Due to GUP correction the allowed energy bands, i.e. AB1 and AB2 (fig.1) are situated within a finite region. If a particle from region (I) encounter a forbidden band (say FB2) in region (II) (fig.3a,3b), then we can bring it to an allowed energy band (AB1) by raising the potential (fig.3c)¹. If we further increase the potential we will find the particle again to be in a forbidden energy band (FB1) (fig.3d) which does not occur in Dirac theory without GUP. The particle would not be able to penetrate the barrier on further increment of potential.

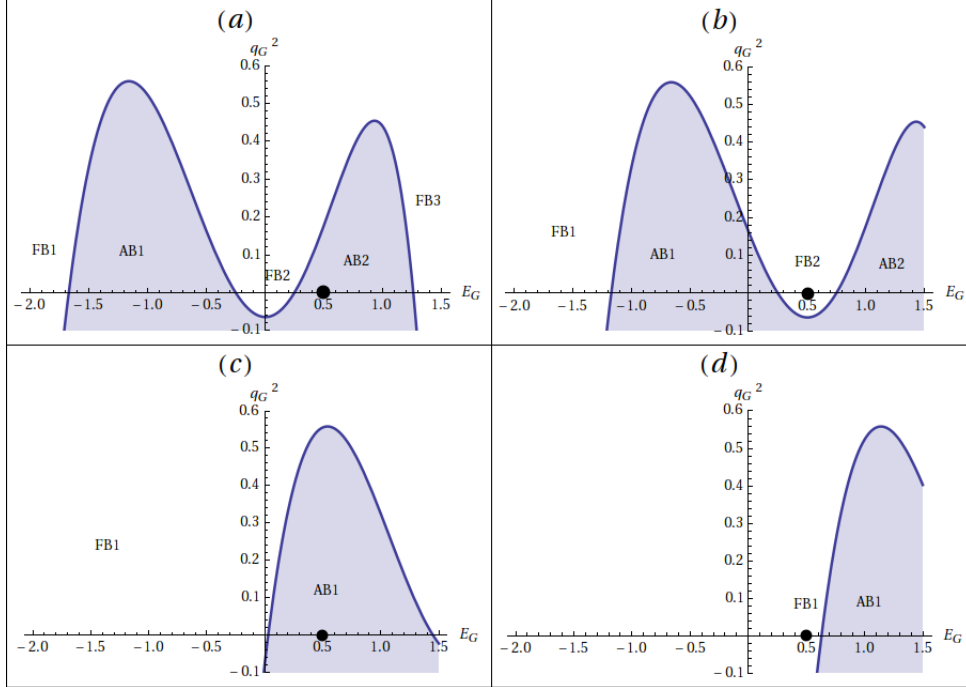


Figure 3: (a) A free particle (black dot) in an allowed band (AB2) ($E_G = 0.5E_P$, $V_0 = 0$). (b) In a potential step the energy spectra will shift by an amount equal to the step height and the particle will find itself in a forbidden band (FB2) ($V_0 = 0.5E_P$). (c) By raising the potential energy, the spectrum can be shifted further to the left which can bring the particle again in an allowed band (AB1) ($V_0 = 1.7E_P$). (d) If the potential is further increased then at a point the particle will again fall in a forbidden band (FB1) ($V_0 = 2.3E_P$).

5 GUP, massless particle and (reverse) Klein paradox

In this section we will study the behaviour of a massless particle inside a potential step. The solution for the modified Dirac equation will be similar to the ansatz (36). We can write it as

$$\psi_{tr}^{G_0} = e^{iq_{G_0}z} \begin{pmatrix} \chi \\ \mathcal{U}'_{G_0} \sigma_z \chi \end{pmatrix} \quad (39)$$

¹An additional constant positive potential will shift the entire energy spectrum to the left and thus can bring the particle to a forbidden band from an allowed band.

where q_{G_0} can be found by putting $m = 0$ in (37) and is given by

$$\hbar^2 q_{G_0}^2 = \frac{(E_G - V_0)^2}{c^2 - 2a(E_G - V_0)c} \left[1 + \frac{(a^2 - 2b)(E_G - V_0)^2}{4(c - 2a(E_G - V_0))^2} \right] \quad (40)$$

Expression for the wave vector in free space (k_{G_0}) can easily be found from (40) by putting $V_0 = 0$.

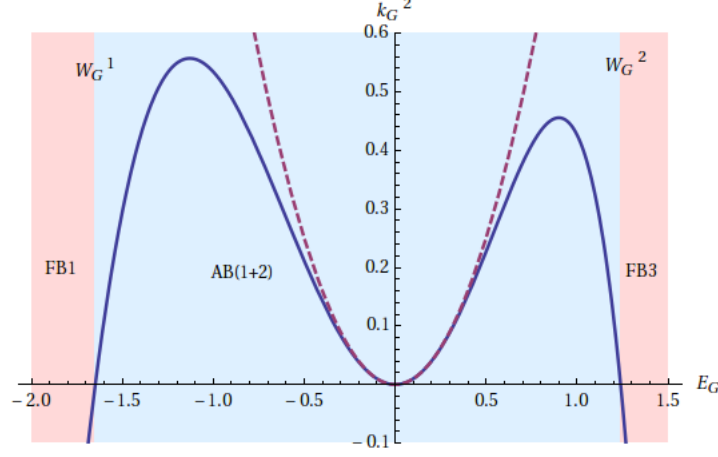


Figure 4: $k_{G_0}^2$ as a function of energy for a massless Dirac particle for $a = 0.05$ and $b = 1$. There is no forbidden energy band around the zero energy, but the forbidden energy bands on the both ends still exist (FB1 and FB3). The dashed line shows the dispersion relation in absence of GUP.

In a non GUP Dirac theory, there is no such thing as Klein paradox for a massless particle due to the absence of any forbidden band. A massless Dirac particle in a non GUP framework can penetrate a potential barrier of any height. In presence of a GUP we see that (*fig.4*) there exist forbidden bands (FB1 and FB3) on the both sides of an allowed band (AB(1+2)). By applying a sufficiently large potential ($\sim E_P$) we can bring this forbidden region to any energy level which will cause a massless Dirac particle with that energy to bounce from the potential wall. This phenomenon is not like a Klein paradox, rather the reverse of it where due to increase of the potential a particle which can propagate earlier will now reflect from the wall of the potential. This is the consequence of the assumption of existence of a minimum length and a maximum energy.

6 Discussion

In this letter we discussed the modification of the dispersion relation of a Dirac particle and how it affects the band structure of a Dirac particle in presence of a minimum length and maximum energy. Then we studied the Klein paradox with the help of this new band structure. If a Dirac particle with some fixed energy face a forbidden energy band in a potential region, it will rebound from the wall of the potential. By increasing the potential strength we can lift up the negative energy levels such a way that there will be an allowed energy band corresponding to the particle energy and the particle will propagate within the potential region (*fig2*). This is the essence of Klein paradox. In presence of a GUP the scenario is different. We see that there are additional forbidden bands apart from the $\pm m$ region, denoted by a negative k_G^2 in (*fig.1*) which appear at the both ends of the energy spectra. The allowed energy bands are not extended over $\pm\infty$, rather localized within a finite region. This kind of band structure is the reason behind the difference in Klein paradox at high energy from that in case of a non GUP theory where the allowed bands are not bounded. We can initially bring a particle to an allowed band from a forbidden band by raising the potential (like what happen in a non GUP Klein paradox) (*fig.1a, b, c*); but if we keep the potential increasing we will reach the end of the allowed band and the particle will find itself again in a forbidden band (*fig.1d*). As a result it will reflect from the wall of the potential.

We further studied the phenomenon for a massless Dirac particle. In case of a massless Dirac particle there is nothing like a forbidden energy band in relativistic quantum mechanics and hence a massless particle with any energy can penetrate and propagate within a potential region of arbitrary height. In presence of a GUP the situation is different. For zero mass the forbidden region in the middle (FB2 in *fig.1*) disappears but those at the both ends (FB1, FB3) still survive (*fig.4*). So in presence of a sufficiently large potential ($\sim E_P$) it is always possible for a massless particle with any energy to encounter a forbidden band. So with increase of height of the potential, a massless Dirac

particle which earlier can penetrate inside the potential region will bounce off the wall like a massive particle does in nonrelativistic quantum mechanics. This is solely due the boundedness of the allowed energy bands which is a consequence of the existence of a minimum length and maximum energy (momentum). Hence the reflection of a massless Dirac particle from a potential barrier of height of the order of Planck energy is direct a signature of existence of a minimum length and maximum energy.

7 Acknowledgement

The author like to thank Council of Scientific and Industrial Research (CSIR), India for financial support.

References

- [1] G. Amelino-Camelia, Int.J.Mod.Phys. D11 (2002) (35-60) [arXiv:gr-qc/0012051]; Phys. Lett. B 510, 2001, (255-263) [arXiv:hep-th/0012238v1]
- [2] D. Amati, M. Ciafaloni and G. Veneziano, Phys. Lett. B 216, 1989, (41-47)
- [3] M. Maggiore, Phys. Lett. B 319 (1993) 83 [arXiv:hep-th/9309034]
- [4] A. Kempf, G. Mangano, R. B. Mann, Phys. Rev. D 52 (1108-1118) 1995 [arXiv:hep-th/9412167]
- [5] F. Girelli, E. R. Livine and D. Oriti, Nuclear Physics B 708, 2005, (411-433) [arXiv:gr-qc/0406100]
- [6] S. Hossenfelder, Phys. Rev. D 73 (105013), 2006 [arXiv:hep-th/0603032]
- [7] S. Hossenfelder, Class. Quant. Grav. 25 (038003), 2008 [arXiv:0712.2811]
- [8] S. Hossenfelder, Class. Quant. Grav. 23, 2006, (1815-1821) [arXiv:hep-th/0510245]
- [9] G. Amelino-Camelia, M. Arzano, Y. Ling and G. Mandanici, Class. Quant. Grav. 23 (2585-2606) 2006 [arXiv:gr-qc/0506110]
- [10] M. Maggiore, Phys. Lett. B 304 (1993) 65 [arXiv:hep-th/9301067]
- [11] F. Scardigli, Phys. Lett. B 452 (1999) 39 [arXiv:hep-th/9904025]
- [12] Rabin Banerjee, Sumit Ghosh, Phys.Lett.B 688, (224-229) 2010 [arXiv:gr-qc/1002.2302]
- [13] Cosimo Bambi, Class. Quant. Grav.25 (105003) 2008 [arXiv:gr-qc/0804.4746]
- [14] Ahmed Farag Ali, Saurya Das, Elias C. Vagenas, Phys. Lett. B 678 (497-499) 2009 [arXiv:hep-th/0906.5396]
- [15] Saurya Das, Elias C. Vagenas, Ahmed Farag Ali, Phys. Lett. B 690 (407-412) 2010 [arXiv:hep-th/1005.3368]
- [16] Ahmed Farag Ali, Saurya Das, Elias C. Vagenas, Phys. Rev. D 84 (044013) 2011 [arXiv:hep-th/1107.3164]
- [17] Saurya Das, R.B. Mann, [arXiv:hep-th/1109.3258] (to appear in PLB)
- [18] Pouria Pedram, Phys. Lett. B 702 (295-298) 2011
- [19] Zachary Lewis and Tatsu Takeuchi, Phys. Rev. D 84, 105029 (2011) [arXiv:hep-th/1109.2680]